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LETTER TO THE EDITOR

Critical surface for a three-colour site percolation problem on the triangular lattice†

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Abstract. A certain relationship between a recent conjecture by Wu and one by Klein *et al* is established through considering a three-colour site percolation problem, where the three sublattices of a triangular lattice are populated with probability s_1, s_2 and s_3, respectively. Both conjectures imply the same critical condition for various special cases of this three-colour model, including the threshold probability $1/\sqrt{2}$ for the honeycomb site problem. Existing numerical estimates provide strong evidence against this threshold value, hence against both conjectures.

To fill the gaps left by missing exact results, a growing number of conjectures related to various features of the percolation problem have been introduced recently. One of these is a conjecture by Wu (1979) concerning the phase boundary of a q-state Potts model on the triangular lattice, where besides the usual two-body couplings one has also three-body couplings over every triangular face. From this Wu has been able to derive the threshold probability for the Kagomé bond percolation problem in the usual limit $q \to 1$. Now a generalised star-triangle transformation proposed by the present author (Kondor 1980) enables one to set up a whole cycle of transformations (much as in the Ising model (Syozi 1972)) among a number of combined (site-bond or two-site plus three-site bond) percolation problems. Starting from Wu's conjecture, it has thus been possible to derive the critical condition for all these systems.

Another, completely different kind of conjecture is due to Klein *et al* (1978), who put forward arguments that, in the special case of the triangular site problem, the simple one-parameter renormalisation group transformation introduced earlier by Reynolds *et al* (1977) might in fact be exact, along with the value of the connectedness length exponent $\nu = \ln\sqrt{3}/\ln\frac{3}{2}$ it implies. This value has later been challenged by the rival conjecture $\nu = \frac{4}{3}$ due to den Nijs (1979).

The purpose of the present Letter is twofold. Firstly, we show that, somewhat surprisingly, a certain relation does exist between the conjecture of Wu (1979) and that of Klein *et al* (1978), in that they have the same implications for the critical condition of some site problems, including the value $1/\sqrt{2} = 0{\cdot}707$ for the honeycomb site threshold. Secondly, we point out that existing numerical estimates, namely the series expansion estimates of Sykes *et al* (1976) and the more recent Monte Carlo results of Vicsek and

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Kertész (1980, private communication), provide very strong evidence against this threshold value, hence against both the conjectures mentioned above.

To show up the link between the two conjectures, we introduce a coloured site problem on the triangular lattice. The function of colours will be to distinguish between the three sublattices shown in figure 1, whose sites are occupied with probability s_1, s_2, and s_3, respectively. Clusters, connectedness, etc are defined regardless of the colours, so the problem considered here is completely different from the polychromatic percolation introduced by Zallen (1977). In spite of the similarity of figure 1 to the construction in a recent paper by Devoret (1980), the problem studied below is also totally different from his.

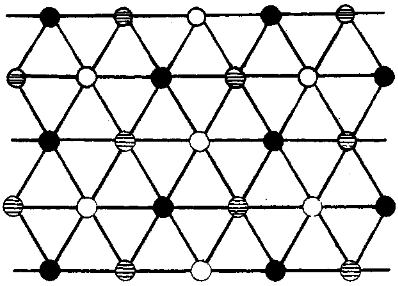

Figure 1. Definition of the three-colour model. The vertices of the three sublattices (black, grey and white) of a triangular lattice are occupied with probability s_i, $i = 1, 2, 3$. Clusters, connectedness and other percolation properties are defined regardless of the colours, so the coloured model is, in fact, a standard site percolation problem with a short-range inhomogeneity.

In particular instances the three-colour problem reduces to various simple site models. Namely, if

$$s_1 = s_2 = s_3, \qquad \text{or } s_i = 0,\ s_j = 1,\ s_k \in (0, 1),\ i, j, k = 1, 2, 3,$$

we are back to a simple triangular site problem. If any of the s_i is equal to zero, we have a honeycomb model with two different site probabilities on the remaining two sublattices. If any of the s_i is equal to unity, we have a model which is easily seen to be equivalent to a site problem on the diced lattice, where all six-coordinated sites are present with probability one, while half the three-coordinated sites are occupied with probability s_j, and the rest with s_k. These special cases are illustrated in figure 2.

Our purpose is now to find out the critical surface of the three-colour triangular model in figure 1. The parameter space of the model is the unit cube $0 \leqslant s_i \leqslant 1$, $i = 1, 2, 3$. The critical surface is given by an equation of the form

$$f(s_1, s_2, s_3) = \tfrac{1}{2} \tag{1}$$

where f is a symmetric function of its arguments due to the equivalence of colours. (The $\frac{1}{2}$ on the RHS is separated off for convenience.) It is also clear that, due to the particular geometry of the triangular lattice, f must be left unchanged by a reflection with respect to the unicolour point $s_1 = s_2 = s_3 = \frac{1}{2}$. To see this, one has essentially to repeat the reasoning which led Sykes and Essam (1964) to the threshold value $s_c = \frac{1}{2}$ for the unicolour problem: following their line of thought, as presented in Essam (1972), one can easily show that in the three-colour problem the average number of clusters (per

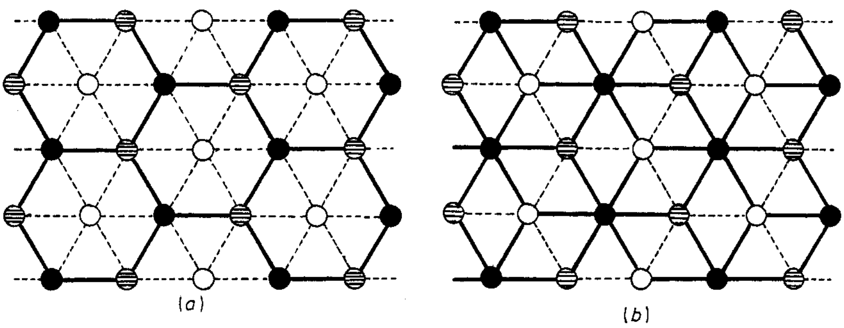

Figure 2. Special cases of the three-colour model. (a) If the vertices of, say, the white sublattice are occupied with probability zero, the (broken) bonds conncting them to the rest of the lattice can be removed without affecting the connectivity on the other two sublattices. The result is the honeycomb lattice. If, in addition, another sublattice, say the black one, is populated with probability unity, we are left with a triangular model on the grey sublattiçe. (b) If, say, the black sublattice is populated with probability unity, the (broken) bonds connecting the vertices of the other two sublattices with each other can be removed without changing the connectivity properties of the system. This leads to the diced lattice. If, in addition, another sublattice, say the white one, is populated with zero probability, we are back to the grey triangular lattice again.

site) $k(s_1, s_2, s_3)$ satisfies the functional equation

$$k(s_1, s_2, s_3) = \phi(s_1, s_2, s_3) + k(1-s_1, 1-s_2, 1-s_3) \tag{2}$$

where

$$\phi = \tfrac{1}{3}(s_1+s_2+s_3) - (s_1s_2+s_1s_3+s_2s_3) + 2s_1s_2s_3,$$

ϕ being a finite polynomial; the symmetry of f (the surface along which k is singular) follows from (2). In the unicolour problem this is all one needs to locate the percolation threshold. In the three-colour problem equation (2) leaves a great deal of freedom. In any case, the permutation and reflection symmetry together imply that for $s_i + s_j = 1$ the critical condition is $s_k = \frac{1}{2}$. This is in accord with what one expects intuitively and it means that the critical surface must contain the straight lines just mentioned.

In order to guess the actual form of equation (1), we shall now make use of the conjecture by Wu (1979) taken in its general form, i.e. with two different three-site couplings over the up-pointing (down-pointing) triangles. From here the application of the generalised star-triangle transformation (Kondor 1980) and of duality takes one to the site-bond problems on the diced and the honeycomb lattice, respectively. Setting all bond probabilities to unity, one finds that the critical condition for the honeycomb site model in figure 2(a) is

$$2s_1s_2 = 1 \tag{3}$$

while for the diced problem in figure 2(b)

$$2(1-s_1)(1-s_2) = 1. \tag{4}$$

Equations (3) and (4) fix the intersections of the surface f with the faces of the unit cube. On these grounds we conjecture the critical condition for the three-colour case:

$$f = s_1s_2 + s_1s_3 + s_2s_3 - 2s_1s_2s_3 = \tfrac{1}{2}. \tag{5}$$

The particular cases mentioned above are readily reproduced from (5). It may be worth comparing (5) with the plane of 'mean probability' $s_1+s_2+s_3=\frac{3}{2}$, which would be the 'critical surface' if the particles were freely distributed over the entire lattice, regardless of the colours. This plane intersects the surface f along the straight lines $s_i=\frac{1}{2}$, $s_j+s_k=1$. A little reflection shows that the relative positions of these two surfaces should also be right; e.g. for $0<s_1, s_2<\frac{1}{2}$ and $\frac{1}{2}<s_3<1$ the surface f must lie above the plane of mean probability, as it does.

The qualitative features of f are therefore certainly correct. On the other hand, we should clearly keep in mind that equation (5) is a conjecture built on another conjecture, basically on that of Wu (1979), so its status is, to say the least, rather shaky. The remarkable fact about (5) is that this same conjecture can be reached from a completely different direction, as we now proceed to show.

Let us recall the simple one-parameter renormalisation group transformation proposed by Reynolds *et al* (1977). When treating the triangular site problem these authors chose the smallest possible triangular cell and applied the majority rule to obtain the cell probability

$$p'=3p^2-2p^3. \tag{6}$$

In a later paper Klein *et al* (1978) argued that for the triangular lattice the simple renormalisation group mapping (6) might, in fact, be exact. Though the situation is at present far from being clear, let us for a moment blindly follow Reynolds *et al* (1977) and transfer their prescription to the three-colour problem. On the scale of the renormalised system the colours are washed away, and the majority rule gives for the cell probability

$$p'=s_1s_2+s_1s_3+s_2s_3-2s_1s_2s_3.$$

The critical probability of the resulting unicolour system being $p'_c=\frac{1}{2}$, this is precisely what equation (5) says!

Thus we have established a certain link between the conjectures of Wu (1979) and Klein *et al* (1978), respectively. The precise logical relationship is this: equations (3) and (4) are direct corollaries of Wu (1979) and are also special cases of equation (5), while the conjecture of Klein *et al* (1978) seems to imply the full equation (5). In particular, for the honeycomb site threshold (which can be obtained from (3) with $s_1=s_2$ or from (5) with $s_1=s_2$, $s_3=0$) both these conjectures lead to $s_c=1/\sqrt{2}=0{\cdot}707$. This can now be compared with available data. Although the agreement with earlier numerical estimates (reviewed in Essam (1972)) may be regarded as fair, it seems as if the data favoured a slightly smaller value. At the given level of uncertainty (roughly one per cent) a final decision cannot yet be made, however. Later, more precise series expansion estimates by Sykes *et al* (1976) yield $s_c=0{\cdot}698\pm0{\cdot}003$, while a very recent Monte Carlo work due to Vicsek and Kertész (1980, private communication) leads to $s_c=0{\cdot}6973\pm0{\cdot}001$. Obviously, there is no way of reconciling these numbers with $1/\sqrt{2}$. We are thus led to the unavoidable conclusion that neither the conjecture of Wu (1979) nor that of Klein *et al* (1978) can be right.

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References

Devoret M 1980 *J. Phys. C: Solid State Phys.* **13** 2257–68

Essam J W 1972 *Phase Transitions and Critical Phenomena* vol 2 ed. C Domb and M S Green (London, New York: Academic) pp 197–270

Klein W, Stanley H E, Reynolds P J and Coniglio A 1978 *Phys. Rev. Lett.* **41** 1145–8

Kondor I 1980 *J. Phys. C: Solid State Phys.* **13** L531–4

den Nijs M P M 1979 *J. Phys. A: Math. Gen.* **12** 1857–68

Reynolds P J, Klein W and Stanley H E 1977 *J. Phys. C: Solid State Phys.* **10** L167–72

Sykes M F and Essam J W 1964 *J. Math. Phys.* **8** 1117–27

Sykes M F, Gaunt D S and Glen M 1976 *J. Phys. A: Math. Gen.* **9** 97–103

Syozi I 1972 *Phase Transitions and Critical Phenomena* vol 1 ed. C Domb and M S Green (London, New York: Academic) pp 269–329

Wu F Y 1979 *J. Phys. C: Solid State Phys.* **12** L645–50

Zallen R 1977 *Phys. Rev.* B **16** 1426–35